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Shifted $1/N$ expansion approach to the interaction $V(r) = r^2 + \lambda r^2/(1+gr^2)$

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Abstract. We use the shifted $1/N$ expansion method to calculate the energy eigenvalues of the non-polynomial oscillator $V(r) = r^2 + \lambda r^2/(1+gr^2)$. The results thus obtained are in excellent agreement with exact numerical results. It is also shown that the non-polynomial interaction can be made supersymmetric and in this case also the results obtained by the shifted $1/N$ method matches extremely well with the exact analytic results given by supersymmetry.

1. Introduction

The shifted $1/N$ expansion method [1, 2] is an extremely powerful method of solving the Schrödinger equation and of late it has been used in a number of problems [3]. The chief merits of this method are: (i) it is non-perturbative in nature and hence can be used in problems which do not necessarily involve small coupling constant; (ii) the method is simple and gives fairly accurate eigenvalues. The shifted $1/N$ expansion differs from the large N expansion [4] in the expression of the expansion parameter. In the former case the expansion parameter is $1/\bar{k}$, $\bar{k} = N + 2l - a$, whereas in the latter case it is $1/k$ where $k = N + 2l$, N is the number of spatial dimensions, $l(l+N-a)\hbar^2$ is the square of the eigenvalue of the N-dimensional orbital angular momentum and a is the shift chosen by requiring agreement between the $1/\bar{k}$ expansion and the exact analytic results for the harmonic oscillator and the Coulomb potential.

In the present paper we shall employ the shifted $1/N$ method to determine the energy eigenvalues of the non-polynomial oscillator represented by the potential $V(r) = r^2 + \lambda r^2/(1+gr^2)$. This interaction is particularly important and its application ranges from purely phenomenological needs of quantum mechanics to field theory in zero dimensions [5] and quantum optics [6]. In particular a number of papers have been devoted to the study of this potential in one dimension [7, 8]. In contrast, relatively little information is available in the three-dimensional case. Znojil [9] has constructed the exact wavefunctions and analytic (continued fraction) Green function in one and three dimensions and also the asymptotic power-series expansions of the Green function by the fixed-point perturbation theory.

As mentioned earlier, the non-polynomial oscillator has been studied by a variety of methods [7-9] and accurate results have been obtained. Here we shall derive some approximate results (by the shifted $1/N$ expansion method) as well as some exact analytical results. To obtain these exact results it will be shown that the non-polynomial interaction is of a supersymmetric nature [10] if the coupling constants satisfy certain relation between them. As soon as we supersymmetrise the interaction analytical

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expansions for various ground-state wavefunctions and energies can be found practically without any calculation. It will be shown that the shifted $1/N$ method gives results which match very well with these exact results and this would be another indicator of the accuracy of our results.

The organisation of the paper is as follows. In § 2 we briefly describe the shifted $1/N$ expansion method and collect the necessary formulae. In § 3 we show that the non-polynomial interaction has a supersymmetric origin and find expressions for the energy values. Section 4 contains the results and finally § 5 is devoted to a discussion.

2. An outline of the shifted $1/N$ expansion method

The radial Schrödinger equation in N spatial dimension is [2]

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{(k-1)(k-3)\hbar^2}{8mr^2}+V(r)\right)\psi(r)=E\psi(r) \tag{1}$$

where

$$\begin{aligned} V(r) &= r^2+\lambda r^2/(1+gr^2) \\ k &= N+2l. \end{aligned} \tag{2}$$

In terms of the shifted variable $\bar{k}=k-a$, one has

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\bar{k}^2[1-(1-a)/\bar{k}][1-(3-a)/\bar{k}]\hbar^2}{8mr^2}+V(r)\right)\psi(r)=E\psi(r). \tag{3}$$

In order to get useful results from a $1/\bar{k}$ expansion, the large E limit of the potential must be suitably defined [2]. The potential $V(r)$ should behave like $\bar{k}^2$ at large $\bar{k}$ since the angular momentum barrier term behaves so. This will give rise to an effective potential which does not vary with $\bar{k}$ at large values of $\bar{k}$, resulting in a sensible zeroth-order classical result. So we consider the following equation [2]:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(r)}{dr^2}+\bar{k}^2\left(\frac{\hbar^2[1-(1-a)/\bar{k}][1-(3-a)/\bar{k}]}{8mr^2}+\frac{V(r)}{Q}\right)\psi(r)=E\psi(r) \tag{4}$$

where Q is a constant to be specified later.

The shifted $1/N$ expansion method consists in solving equation (4) systematically in terms of the expansion parameter $1/\bar{k}$. The leading contribution to the energy comes from the effective potential

$$V_{\text{eff}}(r)=\frac{\hbar^2}{8mr^2}+\frac{V(r)}{Q}. \tag{5}$$

Now it is assumed that $V(r)$ is sufficiently well behaved so that $V_{\text{eff}}(r)$ has a minimum at $r=r_0$ and there are well defined bound states. Then the following relationship is valid:

$$4mr_0^3V^{\mathrm{I}}(r_0)=\hbar^2Q \qquad V^{\mathrm{I}}(r_0)=(dV/dr)|_{r=r_0} \tag{6}$$

where r_0 is the root of the equation

$$\begin{aligned} &N+2l-2+(2n+1)\left(3+\frac{r_0V^{\mathrm{II}}(r_0)}{V^{\mathrm{I}}(r_0)}\right)^{1/2} \\ &\quad=\left(\frac{4mr_0^3V^{\mathrm{I}}(r_0)}{\hbar^2}\right)^{1/2} \qquad V^{\mathrm{II}}(r_0)=(d^2V/dr^2)|. \end{aligned} \tag{7}$$

(The derivation of equation (7) is given in [2].)

Once r_0 is determined, the leading term in E is given by

$$\bar{k}^2 V_{\text{eff}}(r_0) = \frac{\bar{k}^2}{r_0^2}\left(\frac{\hbar^2}{8m} + \frac{r_0^2 V(r_0)}{Q}\right). \tag{8}$$

The next contribution is of order $\bar{k}$ and it is [2]:

$$\frac{\bar{k}}{r_0^2}\left[\left(n+\frac{1}{2}\right)\hbar w - \frac{(2-a)\hbar^2}{4m}\right]. \tag{9}$$

The shift a is chosen so that this contribution vanishes. Therefore we get

$$a = 2 - 2(2n+1)mw/\hbar. \tag{10}$$

The successive contributions to the energy are determined by the substitution

$$x = \bar{k}^{1/2}(r - r_0)/r_0 \tag{11}$$

in equation (4) and expanding about $x = 0$ in powers of x. We are presenting here only the essential steps because the algorithm for the shifted $1/N$ expansion has been developed previously [2].

The energy values are given by expansion in powers of $1/\bar{k}$ where $\bar{k} = n + 2l - a$ as:

$$E_{n,l} = \frac{\bar{k}}{r_0^2}\left[\frac{\hbar^2\bar{k}}{8m} + \frac{r_0^2\bar{k}V(r_0)}{Q} + \frac{\beta^{(1)}}{\bar{k}} + \frac{\beta^{(2)}}{\bar{k}^2} + \mathrm{O}\left(\frac{1}{\bar{k}^3}\right)\right] \tag{12}$$

where

$$\beta^{(1)} = \frac{\hbar^2(1-a)(3-a)}{8m} + (1+2n)\tilde{\varepsilon}_2 + 3(1+2n+2n^2)\tilde{\varepsilon}_4 - 1/\hbar w$$
$$\times[\tilde{\varepsilon}_1^2 + 6(1+2n)\tilde{\varepsilon}_1\tilde{\varepsilon}_3 + (11+30n+30n^2)\tilde{\varepsilon}_3^2] \tag{13}$$

$$\beta^{(2)} = (1+2n)\tilde{\delta}_2 + 3(1+2n+2n^2)\tilde{\delta}_4 + 5(3+8n+6n^2+4n^3)\tilde{\delta}_6$$
$$-[(1+2n)\tilde{\varepsilon}_2^2 + 12(1+2n+2n^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4 + 2(21+59n+51n^2+34n^3)\tilde{\varepsilon}_4^2$$
$$+2\tilde{\varepsilon}_1\tilde{\delta}_1 + 6(1+2n)\tilde{\varepsilon}_1\tilde{\delta}_3 + 30(1+2n+2n^2)\tilde{\varepsilon}_1\tilde{\delta}_5 + 6(1+2n)\tilde{\varepsilon}_3\tilde{\delta}_1$$
$$+2(11+30n+30n^2)\tilde{\varepsilon}_3\tilde{\delta}_3 + 10(13+40n+42n^2+28n^3)\tilde{\varepsilon}_3\tilde{\delta}_5]/\hbar w$$
$$+[4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2 + 36(1+2n)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3 + 8(11+30n+30n^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2 + 24(1+2n)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4$$
$$+8(31+78n+78n^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4 + 12(57+189n+225n^2+150n^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4]/(\hbar w)^2$$
$$-[8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3 + 108(1+2n)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2 + 48(11+30n+30n^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3$$
$$+30(31+109n+141n^2+94n^3)\tilde{\varepsilon}_3^4]/(\hbar w)^3. \tag{14}$$

$$\tilde{\varepsilon}_j = \frac{\varepsilon_j}{(2mw|\hbar)^{j/2}} \qquad \tilde{\delta}_j = \frac{\delta_j}{(2mw|\hbar)^{j/2}} \qquad j = 1, 2\ldots \tag{15}$$

$$w = \frac{\hbar}{2m}\left(3 + \frac{r_0 V^{\mathrm{II}}(r_0)}{V^{\mathrm{I}}(r_0)}\right)^{1/2} \tag{16}$$

$$\varepsilon_1 = (2-a)\hbar^2/2m \tag{17}$$

$$\varepsilon_2 = -3\hbar^2(2-a)/4m \tag{18}$$

$$\varepsilon_3 = -\hbar^2/2m + r_0^5 V^{\mathrm{III}}(r_0)/6Q \tag{19}$$

$$\varepsilon_4 = 5\hbar^2/8m + r_0^6 V^{IV}(r_0)/24Q \tag{20}$$

$$\delta_1 = -(1-a)(3-a)\hbar^2/4m \tag{21}$$

$$\delta_2 = 3(1-a)(3-a)\hbar^2/8m \tag{22}$$

$$\delta_3 = (2-a)\hbar^2/m \tag{23}$$

$$\delta_4 = -5(2-a)\hbar^2/4m \tag{24}$$

$$\delta_5 = -3\hbar^2/4m + r_0^7 V^{V}(r_0)/120Q \tag{25}$$

$$\delta_6 = 7\hbar^2/8m + r_0^8 V^{VI}(r_0)/720Q. \tag{26}$$

3. Supersymmetric character of the interaction $V(r) = r^2 + \lambda r^2/(1+gr^2)$ and analytical expressions for the energy eigenvalues

Before showing that the non-polynomial oscillator represents a supersymmetric interaction, let us recall some essential features of supersymmetric quantum mechanics (SUSYQM) in one dimension [12]. In one dimension the Hamiltonian of SUSYQM is given by

$$H^S = \{Q^+, Q\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{27}$$

$$H_\pm = -\mathrm{d}^2/\mathrm{d}x^2 + V_\pm(x) \tag{28}$$

$$V_\pm(x) = W^2(x) \pm W'(x). \tag{29}$$

Here $W(x)$ is called the superpotential and Q and Q^+ are the supercharges whose explicit forms are

$$Q = (p + \mathrm{i}W)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{30}$$

$$Q^+ = (p - \mathrm{i}W)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{31}$$

The relations obeyed by Q, Q^+ and H^S are given by

$$[H^S, Q] = [H^S, Q^+] = 0 \tag{32}$$

$$Q^2 = Q^{+2} = 0. \tag{33}$$

The eigenstates of H^S are of the form

$$\varphi^n(x) = \begin{pmatrix} \varphi_+^n(x) \\ \varphi_-^n(x) \end{pmatrix}. \tag{34}$$

If supersymmetry is unbroken, then the ground-state energy vanishes and the ground-state wavefunctions are of the form

$$\begin{pmatrix} \varphi_+^0(x) \\ 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 \\ \varphi_-^0(x) \end{pmatrix}.$$

The choice between the two ground-state wavefunctions is dictated by the normalisability of $\varphi_\pm^0(x)$ respectively. The forms of $\varphi_\pm^0(x)$ can also be determined easily. To do this we recall if $|\psi\rangle$ is a ground state then

$$Q|\psi\rangle = Q^+|\psi\rangle = 0. \tag{35}$$

From (30) and (31) we then find

$$\varphi_{\pm}^{0}(x) \sim \exp\left(\pm \int^{x} W(t)\, dt \right). \tag{36}$$

To show that the non-polynomial oscillator is supersymmetric we have to chose a suitable superpotential and to this end we take

$$W(r) = \mu r + \frac{2gr}{1+gr^2} + \frac{\nu}{r}. \tag{37}$$

With the choice (37) the supersymmetric potential corresponding to the fermionic sector (+) can easily be found from (29):

$$V_{+}(r) = \mu^2 r^2 + \frac{2g - 4\mu + 4\nu g}{(1+gr^2)} + \frac{\nu(\nu-1)}{r^2} + (2\mu\nu + 5\mu). \tag{38}$$

Note that we can write the potential corresponding to the non-polynomial oscillator in the following form:

$$V(r) = r^2 + \frac{\lambda}{g} - \frac{\lambda/g}{(1+gr^2)}. \tag{39}$$

Therefore, the effective potentials appearing in the radial Schrödinger equation corresponding to (39) and the Schrödinger equation corresponding to (38) are given respectively by

$$V^{\text{eff}}(r) = r^2 - \frac{\lambda/g}{(1+gr^2)} + \frac{l(l+1)}{r^2} \tag{40}$$

$$V_{+}^{\text{eff}}(r) = \mu^2 r^2 + \frac{2g - 4\mu + 4\nu g}{(1+gr^2)} + \frac{\nu(\nu-1)}{r^2}. \tag{41}$$

Hence we can identify the two Schrödinger equations with the effective potentials provided

$$\lambda = -(2g^2 - 4\mu g + 4\nu g^2) \qquad \mu = \pm 1;\ \nu = l+1. \tag{42}$$

The relation between the energy eigenvalues is given by

$$E_l^n - \lambda/g = E_{+}^{n} - 2\mu\nu - 5\mu. \tag{43}$$

From (36) and (37) the ground-state wavefunction corresponding to the fermionic sector (+) is found to be

$$\varphi_{+}^{0}(r) \sim \exp\left(\int^{r} W(t)\, dt \right) = r^{l+1}(1+gr^2) \exp(\mu r^2/2). \tag{44}$$

However, (44) will be an acceptable ground state only if $\varphi_{+}^{0}(r)$ satisfies proper boundary conditions, i.e.

$$\begin{aligned} &\lim_{r\to\infty} \varphi_{+}^{0}(r) = 0 \\ &\lim_{r\to 0} \varphi_{+}^{0}(r) = 0. \end{aligned} \tag{45}$$

(We point out that even though we described SUSYQM in one dimensions it can be applied to the problem at hand only if we remain on the half-line $(0, \infty)$; in that case

we can identify one-dimensional SUSYQM with the three-dimensional radial problem. In SUSYQM this amounts to putting up a barrier at the origin [13].) Equation (44) tells us that $\varphi_+^0(r)$ will be normalisable if $\mu = -1$ and $l > 0$ and so

$$\varphi_+^0(r) \sim r^{l+1}(1+gr^2)\exp(-r^2/2). \tag{46}$$

In this case we have

$$E_+^0 = 0 \tag{47}$$

so that the ground-state energy of the non-polynomial oscillator is given by

$$E_l^0 = \lambda/g + 2(l+1) + 5. \tag{48}$$

It is interesting to note that for each value of l we have a new superpotential, i.e. a new SUSYQM system, but in each case the ground state of SUSYQM is related to the $n = 0$ ground state of the non-polynomial oscillator according to (48). Now E_l^0 can be obtained for various values of λ and g, chosen according to

$$\lambda = -[2g^2 + 4g + 4(l+1)g^2]. \tag{49}$$

In the next section we shall compare ground-state energies found by the shifted $1/N$ expansion method with the exact values given by (48).

4. Results

For potential (2) the successive derivatives are given by the following:

$$V^{\mathrm{I}}(r) = 2r + 2\lambda r/(1+gr^2)^2 \tag{50}$$

$$V^{\mathrm{II}}(r) = 2 + 2\lambda/(1+gr^2)^2 - 8\lambda gr^2/(1+gr^2)^3 \tag{51}$$

$$V^{\mathrm{III}}(r) = 24g\lambda(gr^2-1)/(1+gr^2)^4 \tag{52}$$

$$V^{\mathrm{IV}}(r) = 24g\lambda(10gr^2 - 5g^2r^4 - 1)/(1+gr^2)^5 \tag{53}$$

$$V^{\mathrm{V}}(r) = 24g^2r(30 + 30g^2r^4 - 100gr^2)\lambda/(1+gr^2)^6 \tag{54}$$

$$V^{\mathrm{IV}}(r) = 720\lambda g^2(1 + 35g^2r^4 - 21gr^2 - 7g^3r^6)/(1+gr^2)^7. \tag{55}$$

For any given choice of n and l, equation (7) becomes a transcendental equation when $V^{\mathrm{I}}(r)$ and $V^{\mathrm{II}}(r)$ are substituted into it from equations (50) and (51). This transcendental equation can be solved numerically to obtain r_0. Substitution of r_0 into equations (6), (12)-(26) and (52)-(55) immediately gives the energy eigenvalues. In all our calculations we have taken $\hbar = 2m = 1$.

In table 1 we list our second-order energy values calculated from equation (12) corresponding to $l = -1, 0$ (for comparison with the one-dimensional case [8]) and compare them with those given by Bessis and Bessis [14] and Lai *et al* [15] wherever possible. As is evident from table 1 the agreement is excellent in almost all the cases, the exception being the ground-state values at ($\lambda = 1$, $g = 1$), ($\lambda = 10$, $g = 10$), ($\lambda = 100$, $g = 100$) and ($\lambda = 0.1$, $g = 2$) and the ground-state and second excited state at ($\lambda = 100$, $g = 10$). In these cases the higher-order correction terms are essential because we found that the first-order correction gives better agreement than the second-order correction.

In table 2, the shifted $1/N$ expansion results (with second-order correction) have been compared with the exact supersymmetric results given by (48) and (49). The maximum error obtained in these cases is 0.02%. Finally, we have given the energy values for 2p and 3s-3d states for various values of λ and g in table 3.

Table 1. The first four energy levels for different values of λ and g calculated from the shifted $1/N$ expansion method. The numbers in the round and square brackets correspond to the variational results of Bessis and Bessis [14] and the Padé approximation results of Lai *et al* [15] respectively.

	λ				
	0.1	0.5	1	10	100
$g = 0.1$	1.043 140	1.202 990	1.380 501	3.250 165	9.976 142
	(1.043 174)	(1.203 040)	(1.380 532)	(3.250 261)	(9.976 180)
	[1.043 173]	[1.203 039]	[1.380 532]	[3.250 261]	[9.976 180]
	3.120 039	3.570 678	4.079 807	9.618-83	29.781 198
	(3.120 082)	(3.570 809)	(4.079 883)	(9.619 066)	(29.781 191)
	[3.120 081]	[3.570 809]	[4.079 883]	[9.619 066]	[29.781 191]
	5.181 319	5.872 099	6.668 245	15.728 957	49.292 716
	(5.181 095)	(5.871 584)	(6.667 919)	(15.729 336)	(49.292 691)
	[5.181 094]	[5.871 583]	[6.667 919]	[15.729 336]	[49.292 690]
	7.231 179	8.122 420	9.167 067	21.590 711	68.512 9918
	(7.231 010)	(8.121 871)	(9.166 567)	(21.591 007)	(68.513 052)
	[7.231 009]	[8.121 871]	[9.166 567]	[21.591 005]	[68.513 062]
$g = 0.5$	1.028 982	1.144 596	1.285 758	3.016 427	9.692 159
	(1.031 215)	(1.151 564)	(1.292 951)	(3.016 854)	(9.692 158)
	[1.031 214]	[1.151 563]	[1.292 950]	[3.016 854]	[9.692 157]
	3.074 019	3.363 617	3.712 292	8.481 044	28.362 637
	(3.073 903)	(3.363 801)	(3.713 902)	(8.482 271)	(28.362 598)
	[3.073 902]	[3.363 801]	[3.713 902]	[8.482 270]	[28.362 597]
	5.090 873	5.460 438	5.930 119	12.940 612	45.632 923
	(5.093 069)	(5.463 214)	(5.920 632)	(12.948 033)	(45.636 573)
	[5.093 060]	[5.463 211]	[5.920 632]	[12.948 038]	[45.636 572]
	7.103 912	7.520 581	8.043 514	16.683 664	61.573 534
	(7.105 850)	(7.527 881)	(8.052 379)	(16.679 365)	(61.577 873)
	[7.105 850]	[7.527 881]	[8.052 378]	[16.679 411]	[61.577 873]
$g = 1$	1.021 216	1.099 451	1.198 184	2.777 245	9.359 312
	(1.024 187)	(1.118 589)	(1.232 372)	(2.782 330)	(9.359 418)
	[1.024 123]	[1.118 552]	[1.232 353]	[2.782 330]	[9.359 418]
	3.051 999	3.258 369	3.512 157	7.405 425	26.706 230
	(3.051 651)	(3.255 842)	(3.507 421)	(7.417 506)	(26.705 965)
	[3.051 526]	[3.255 802]	[3.507 397]	[7.417 506]	[26.705 965]
	5.052 327	5.264 521	5.537 385	10.811 607	41.419 437
	(5.059 287)	(5.295 063)	(5.589 861)	(10.701 026)	(41.441 010)
	[5.058 990]	[5.294 916]	[5.589 833]	[10.704 480]	[41.441 099]
	7.062 590	7.312 734	7.625 135	13.495 879	53.820 199
	(7.065 498)	(7.324 540)	(7.648 317)	(13.388 324)	(53.839 093)
	[7.064 966]	[7.234 540]	[7.649 068]	[13.390 003]	[53.839 092]
$g = 10$	1.005 731	1.029 354	1.060 918	1.693 636	5.308 762
	(1.005 943)	(1.029 685)	(1.059 297)	(1.580 025)	(5.793 947)
	3.008 751	3.043 796	3.087 574	3.873 211	11.611 174
	(3.008 811)	(3.044 051)	(3.088 091)	(3.879 037)	(11.572 198)
	5.008 275	5.041 422	5.082 822	5.824 408	13.993 148
	(5.008 280)	(5.041 412)	(5.082 848)	(5.832 769)	(13.628 79)

Table 1. (continued)

	λ				
	0.1	0.5	1	10	100
	7.009 182	7.045-41	7.091 875	7.918 053	16.102 810
	(7.009 038)	(7.045 187)	(7.090 270)	(7.903 155)	(15.988 706)
$g = 100$	1.000 755	1.003 778	1.007 554	1.075 263	1.718 594
	(1.000 841)	(1.004 205)	(1.016 820)	(1.084 064)	(1.836 385)
	3.000 985	3.004 923	3.009 844	3.098 414	3.984 322
	(3.000 983)	(3.004 916)	(3.019 664)	(3.098 317)	(3.983 099)
	5.000 979	5.004 897	5.009 791	5.097 876	5.978 945
	(5.000 926)	(5.004 638)	(5.018 551)	(5.092 762)	(5.928 353)
	7.000 992	7.004 959	7.009 911	7.099 078	7.990 810
	(7.000 985)	(7.004 922)	(7.019 690)	(7.098 449)	(7.984 445)

Table 2. Comparison of the ground-state energy values calculated from the shifted $1/N$ expansion with the exact analytic SUSY results.

			Ground-state energy value	
g	l	λ	$1/N$ expansion	Exact SUSY
0.1	0	−0.46	2.400 520	2.4
0.1	1	−0.5	4.000 116	4.0
0.1	2	−0.54	5.599 965	5.6
0.01	0	−0.0406	2.399 989	2.4
0.01	1	−0.041	4.899 974	4.9
0.1	−1	−0.42	0.801 177	0.80†

† This result has also been obtained by Fack *et al* in [8].

In table 4, we present energy values corresponding to $\lambda = 0.1$, $g = 2$ obtained by the present method, by Bessis and Bessis [14], by Lai *et al* [15] and Handy [8] together with the percentage deviations from Handy's result. It is seen from table 3 that the energy values corresponding to the first excited state obtained by the present method are in better agreement with Handy's result [8] (by comparison with values obtained by Bessis and Bessis [14] and Lai *et al* [15]) though the ground-state result is less accurate, as can be expected in the $1/N$ formalism.

5. Discussion

In this paper we have applied the shifted $1/N$ expansion to find energy eigenvalues of the non-polynomial oscillator $V(r) = r^2 + \lambda r^2/(1 + gr^2)$. The results we have found are in excellent agreement with accurate results of [14, 15]. Furthermore, it has been shown that the non-polynomial interaction has a supersymmetric character provided the coupling constants satisfy a certain relation between them. In this case, using the SUSY identification, ground states can be found almost trivially. The accuracy of the shifted $1/N$ method has also been tested in a different way: we have calculated the

Table 3. The energy values for 2p, 3s-3d states for various values of λ and g.

		g				
λ	State	0.1	0.5	1	10	100
0.1	2p	5.186 338	5.100 976	5.065 610	5.009 406	5.000 994
	3s	11.309 171	11.115 476	11.067 785	11.009 416	11.000 993
	3p	9.276 635	9.117 522	9.070 745	9.009 504	9.000 999
	3d	7.243 927	7.119 005	7.073 713	7.009 622	7.000 994
0.5	2p	5.893 494	5.500 470	5.327 161	5.047 058	5.004 968
	3s	12.519 642	11.578 069	11.338 469	11.047 083	11.004 970
	3p	10.350 311	9.586 705	9.353 303	9.047 530	9.004 970
	3d	8.177 754	7.592 014	7.368 046	7.048 123	7.004 982
1	2p	6.704 090	5.989 770	5.652 112	5.094 107	5.009 935
	3s	13.969 894	12.158 447	11.675 801	11.094 163	11.009 936
	3p	11.622 346	10.171 198	9.705 584	9.095 056	9.009 948
	3d	9.261 812	8.176 220	7.734 778	7.096 239	7.009 960
10	2p	15.813 628	13.244 687	11.071 273	5.940 689	5.099 346
	3s	32.609 516	23.005 401	17.702 350	11.941 423	11.099 363
	3p	27.293 083	20.125 410	15.962 474	9.950 350	9.099 463
	3d	21.836 043	17.394 651	14.089 958	14.089 958	7.962 262
100	2p	49.389 615	46.083 221	42.236 452	14.363 739	5.993 565
	3s	106.088 741	89.718 266	72.718 487	20.389 517	11.993 716
	3p	87.539 397	76.654 728	64.802 034	18.482 766	9.994 694
	3d	68.801 562	62.857 148	55.976 148	16.611 028	7.996 048

Table 4. Energy values corresponding to the ground state and the first excited state when $\lambda = 0.1$ and $g = 2$.

	Ground state		First excited state	
	E_0	Deviation from Handy [8]	E_1	Deviation from Handy [8]
Roy *et al*	1.019 046†	0.182%	3.032 875	0.003%
Bessis and Bessis [14]	1.017 894	0.009%	3.031 773	0.006%
Lai *et al* [15]	1.017 281	0.069%	3.032 957	0.032%
Handy [8]	1.017 185		3.032 772	

† Only first-order correction has been taken.

same states, as those given by SUSY, by the shifted $1/N$ method and these energy values match the exact SUSY values extremely well. In fact it can be shown that SUSY provides a check for the numerical calculation of eigenvalues for many well known potentials for which analytical results cannot be obtained otherwise. Further work in this direction is in progress.

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